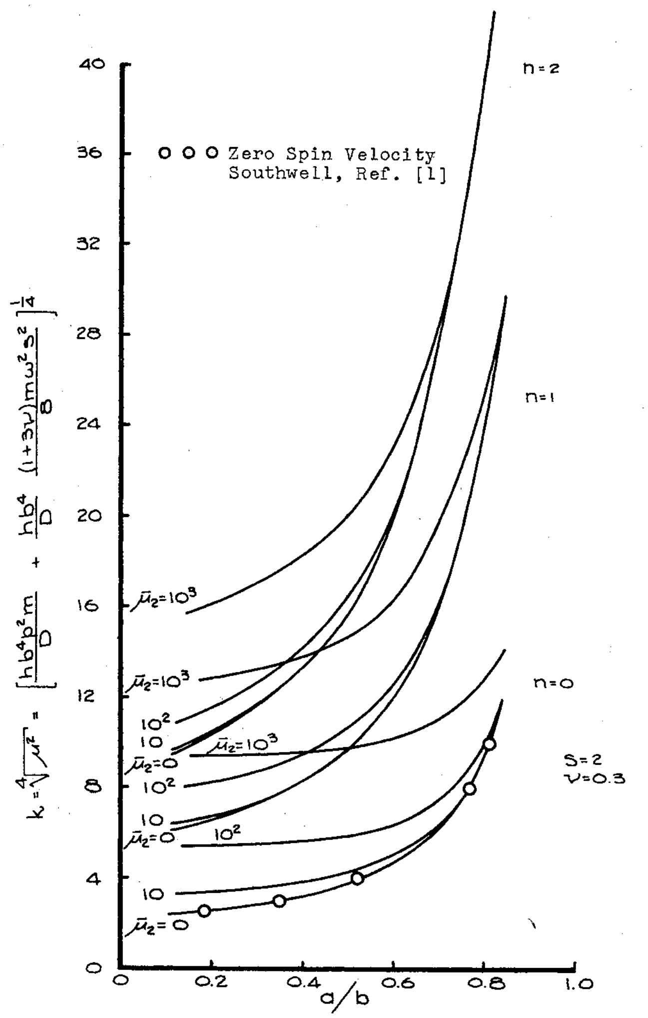

Fig. 4 Fourth root of the frequency parameter μ^2 as a function of the hub to disk radius ratio for two nodal diameters and from zero to two nodal circles.

both the present analysis and from Southwell's[2] work and the agreement is exact to within acceptable computational errors. The cases of large $\bar{\mu}_2$ were compared with Eversman's[8] work and it was found that for $\bar{\mu}_2 \to \infty$ the frequency curves approached those of the membrane. However, carrying this limiting process to extremely high values of $\bar{\mu}_2$ is difficult because the differential equations and boundary conditions obtained in the limit are of different character than those for finite values of $\bar{\mu}_2$.

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Bounds on the Motion of Objects Ejected from an Orbiting Body

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Introduction

THE purpose of this Note is to present a procedure for determining bounds on the position of objects ejected from an orbiting body or a space vehicle relative to the vehicle trajectory's position. The results are valid for any velocity and direction of ejection. The same analysis can be used to determine an estimate of the predicted vehicle trajectory for a radar observation velocity error in an arbitrary direction. Both problems are described by the same set of linear differential equations with varying coefficients. However, these equations cannot be reduced to a closed form of solution. As a result, the usual analysis has been either to replace the varying coefficients by their average constant values or to make use of computer solutions. In this Note, reasonable bounds on the object's position are obtained by the use of an integral inequality without approximating the solution of the exact or averaged differential equations.

Equations of Motion

The motion of an object in an inverse square gravitational field is given by the differential equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \cos^2\phi - r\left(\frac{d\phi}{dt}\right)^2 = \frac{-g_0R_0{}^2}{r^2} \qquad (1)$$

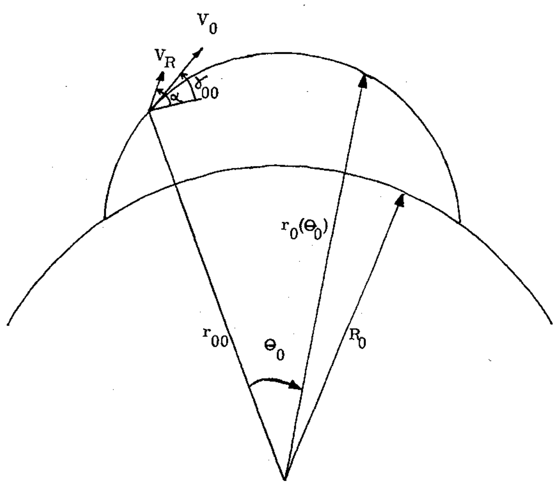

Fig. 1 Trajectory diagram.

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$$\frac{1}{r\cos\phi}\frac{d}{dt}\left(r^2\cos^2\phi\frac{d\theta}{dt}\right) = 0 \quad (2)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) + r\left(\frac{d\theta}{dt}\right)^2\sin\phi\cos\phi = 0 \quad (3)$$

where ϕ is the angle between r and its projection in the θ plane. In this plane, r_0, θ_0 are the polar coordinates of the trajectory as shown in Fig. 1. For an object whose motion lies in a plane, $\phi = 0$ and Eq. (3) vanishes whereas (1) and (2) can be solved to give

$$\frac{1}{r_0} = \frac{R_0^2g_0}{C_0^2} + \left(\frac{1}{r_{00}} - \frac{R_0^2g_0}{C_0^2}\right)\cos\theta_0 - \frac{\tan\gamma_{00}}{r_{00}}(\sin\theta_0) \quad (4)$$

where γ_{00} represents the direction with respect to the local horizontal (at $r_0 = r_{00}$) of the object traveling with a velocity V_0; C_0 represents the angular momentum per unit mass for the trajectory; R_0 is the radius of the earth; and g_0 is the gravitational constant.

A particle is then ejected from the object with a velocity V_R in the plane of the trajectory making an angle α with local horizontal and component V_D normal to the trajectory plane. The motion of the ejected particle can be described by $\Delta r = r - r_0$, $\Delta\theta = \theta - \theta_0$, $\Delta\phi = \phi$, $\Delta C = C - C_0$ relative to the motion of the object, where these quantities are substituted into (1), (2), and (3) to give

$$\frac{d^2\Delta r}{dt^2} - \left[\left(\frac{d\theta_0}{dt}\right)^2 + \frac{2g_0R_0^2}{r_0^3}\right]\Delta r - \left(2r_0\frac{d\theta_0}{dt}\right)\frac{d\Delta\theta_0}{dt} = 0 \quad (5)$$

$$r_0^2\, d\Delta\theta/dt + 2r_0(d\theta_0/dt)\Delta r = \Delta C \quad (6)$$

$$(dt/d)\,(r_0^2\, d\Delta\phi/dt) + r_0^2(d\theta_0/dt)^2\,\Delta\phi = 0 \quad (7)$$

where second-order effects have been neglected. Equations (5–7) can be simplified to give[1]

$$\frac{d^2\bar{y}}{d\theta_0^2} - \left[\frac{3g_0R_0^2}{C_0^2}r_0(\theta_0) - 4\right]\bar{y} = \frac{2\Delta C}{C_0} \quad (8)$$

$$d\bar{x}/d\theta_0 + 2\bar{y} = \Delta C/C_0 \quad (9)$$

$$d^2\Delta\phi/d\theta_0^2 + \Delta\phi = 0 \quad (10)$$

where $\bar{x} = (x/r_0) = \Delta\theta$, $\bar{y} = (y/r_0) = (\Delta r/r_0)$, x and y are horizontal and vertical displacements of the particle from the object, and the independent variable t has been changed to θ_0 with the use of $r_0^2\, d\theta_0/dt = C_0$. Equation (10) is independent of (8) and (9) and can be easily solved with the given initial conditions

$$\theta_0 = 0,\ \Delta\phi = 0,\ \frac{d\Delta\phi}{d\theta_0} = \frac{V_D}{V_0\cos\gamma_{00}}\sin\theta_0 \quad (11)$$

and $r_0\Delta\phi = Z$, which is the normal displacement to the trajectory plane. The initial conditions for (8) and (9) are

$$\bar{x} = 0,\ \bar{y} = 0,\ d\bar{y}/d\theta_0 = (\Delta C/C_0)\tan\alpha$$
$$\Delta C = r_{00}V_R\cos\alpha,\ C_0 = r_{00}V_0\cos\gamma_{00} \quad (12)$$

As mentioned earlier, the usual procedure to solve (8) and (9) is by replacing the variable coefficient in (8) by its average value. In this Note, bounds on the solutions of (8) and (9) are obtained without approximating the solution of the exact or averaged equations.

Bounds on Solutions

The solution of Eq. (8) can be represented by the integral equation

$$\bar{y}(\theta_0) = C_1\sin2\theta_0 + C_2\cos2\theta_0 + \frac{1}{2}\int_0^{\theta_0}[\sin2(\theta_0 - \tau)]\left[\frac{3g_0R_0^2}{C_0^2}r_0(\tau)\bar{y}(\tau) + \frac{2\Delta C}{C_0}\right]d\tau \quad (13)$$

where C_1 and C_2 are constants, r_0 is given by (4). Using the initial conditions (12) and performing part of the integration in (13), the equation is simplified to

$$\bar{y}(\theta_0) = \frac{V_R\sin(\alpha + \theta_0)\sin\theta_0}{V_0\cos\gamma_{00}} + \frac{3}{2}\int_0^{\theta_0}\frac{g_0R_0^2}{C_0^2}r_0(\tau)\bar{y}(\tau)\sin2\,(\theta_0 - \tau)d\tau \quad (14)$$

An inequality in $\bar{y}$ can be obtained by taking the absolute value of (14) to give

$$|y(\theta_0)| \leqslant C_3 + \int_0^{\theta_0}C_4(\tau)|\bar{y}(\tau)|\,d\tau \quad (15)$$

where

$$C_3 = [V_R|\sin(\alpha + \theta_0)\sin\theta_0|_{\max}]/(V_0|\cos\gamma_{00}|)$$

for all θ_0 in the range of interest and

$$C_4 = [3g_0R_0^2r_0(\tau)/2C_0^2]|\sin2(\theta_0 - \tau)|$$

Since C_3, $C_4(\tau) > 0$, therefore, by means of the integral inequality,[2]

$$|\bar{y}(\theta_0)| \leqslant C_3\exp\int_0^{\theta_0}C_4(\tau)d\tau = C_3\exp C_5(\theta_0) \quad (16)$$

which gives an upper bound on $|\bar{y}(\theta_0)|$. The integral in (16) may be rewritten as

$$C_5 = \int_0^{\theta_0}\frac{(3/2)\,|\sin2\,(\theta_0 - \tau)|d\tau}{1 + [(C_0^2/g_0R_0^2r_{00}) - 1]\cos\tau - (C_0^2/g_0R_0^2r_{00})\tan\gamma_{00}\sin\tau} \quad (17)$$

For θ_0 in the range $0 < \theta_0 < (\pi/2)$, $|\sin2\,(\theta_0 - \tau)| = \sin2(\theta_0 - \tau)$ and the integral can be evaluated after some simplifications in the denominator to give

$$C_5 = (6/\rho)\sin[(3\theta_0/2) + S]\sin(\theta_0/2) - (3\theta_0/\rho^2)\sin(2\theta_0 + 2S) - (3/\rho^2)\cos(2\theta_0 + 2S)\,In\,[\{1 + \rho\cos(\theta_0 + S)\}/(1 + \rho\cos S)] + [3(2 - \rho^2)/2\rho^2]\sin(2\theta_0 + 2S) \times \begin{cases}[2/(1 - \rho^2)^{1/2}]\tan^{-1}A_1\tan^{-1}A_2 & \rho < 1\\ [1/(\rho^2 - 1)^{1/2}]In[(B_1 + 1)(B_2 - 1)/(B_1 - 1)(B_2 + 1)] & \rho > 1\end{cases} \quad (18)$$

where

$$\rho\sin S = [(V_0^2r_{00})/(2\,g_0R_0^2)]\sin2\gamma_{00}$$

$$\rho\cos S = [(V_0^2\,r_{00})/(2\,g_0R_0^2 - 1)] + (V_0^2r_{00}/2g_0R_0^2)\cos2\gamma_{00}$$

$$A_1 = [(1 - \rho)/(1 + \rho)]^{1/2}\tan(\theta_0 + S)/2,\ A_2 = A_1(\theta_0 = 0)$$

$$B_1 = [(\rho - 1)/(1 + \rho)]^{1/2}\tan(\theta_0 + S)/2,\ B_2 = B_1(\theta_0 = 0)$$

For $\theta_0 > (\pi/2)$, the sign of $\sin2(\theta_0 - \tau)$ must be changed in the appropriate interval and a different result is obtained. The integration (17) would be simplified if $|\sin2\,(\theta_0 - \tau)|$ were replaced by its maximum of one; however, the accuracy of the upper bound on $|\bar{y}|$ would be significantly impaired.

For $\rho = 1$ [$S = 2\gamma_{00}$, $(V_0^2r_{00}/2g_0R_0^2) = 1$] the quantity in the braces of Eq. (18) becomes $\tan[(\theta_0 + S)/2] - \tan(S/2)$ and the corresponding expression for C_5 is reduced. The upper bound on $|y|$ is then obtained by multiplication of (16) by $r_0(\theta_0)$.

In order to obtain a bound on $\bar{x}(\theta_0)$, Eq. (9) can be solved to give

$$\bar{x}(\theta_0) = \frac{\Delta C}{C_0}\,\theta_0 - 2\int_0^{\theta_0} \bar{y}(\tau)d\tau \tag{19}$$

and by means of the triangle inequality

$$\theta_0[|(\Delta C/C_0)| - 2|\bar{y}(\theta_0)|] \leqslant |\bar{x}(\theta_0)| \leqslant \theta_0[|(\Delta C/C_0)| + 2|\bar{y}(\theta_0)|] \tag{20}$$

where $|\bar{y}(\theta_0)|$ is the upper bound obtained in (16). If the upper bound (16) is close to the actual value of $|\bar{y}(\theta)|$, then the lower bound in (20) would give a close approximation to $|\bar{x}(\theta_0)|$. The bound on $|x(\theta)|$ can be obtained by multiplication of (20) by $r_0(\theta_0)$.

The upper bound on $y(\theta_0)$ as given by (16) was found, in some cases, to be more accurate than that obtained from the averaged equations for it produced lower magnitudes. However, the bound on $x(\theta_0)$ was less effective than that of the averaged equations. The reason for the latter discrepancy is that the procedure in this Note produces accurate bounds on $|d\bar{x}/d\theta_0|$, from which bounds on $|\bar{x}(\theta_0)|$ are deduced by a straightforward integration. If $\bar{y}(\theta_0)$ or the variations in $\bar{x}(\theta_0)$ changes sign, the bound on $|\bar{x}(\theta_0)|$ becomes ineffective.

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Correct Formulation of Airfoil Problems in Magnetoaerodynamics

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1. Introduction

In a recent paper[1] Fan and Ludford have dealt with the theory of thin airfoils in magnetoaerodynamics in view of giving a correct solution to this problem. It may be easily observed, however, that the kernels of the integral equations to which the solution of the problem is reduced are expressed by divergent integrals (e.g., $I_{\pm}$). Under such conditions the solution given by the mentioned authors cannot be valid. The same incorrectness is also found in Ref. 2.

The purpose of this Note is to revise the solution of this problem. The method of solution is based essentially on the ideas which have substantiated the first article on this problem[3] and then under a simpler form.[4]

This Note treats only the case of crossed fields, to which the paper by Fan and Ludford also refers.

2. Motion Equations

In dimensionless variables the system of motion equations may be written as follows:

$$M^2\,\partial p/\partial x + \partial v_x/\partial x + \partial v_y/\partial y = 0 \tag{1}$$

$$\partial v_x/\partial x + \partial p/\partial x = S(\partial b_x/\partial y - \partial b_y/\partial x) \tag{2}$$

$$\partial v_y/\partial x + \partial p/\partial y = 0 \tag{3}$$

$$\partial b_x/\partial x + \partial b_y/\partial y = 0 \tag{4}$$

$$\partial b_y/\partial x - \partial b_x/\partial y = R(b_y + v_x) \tag{5}$$

$$\lim_{x^2+y^2\to\infty} (v_x,v_y,p,b_x,b_y) = 0,\ S = A^{-2},\ R = Rm \tag{6}$$

Elimination of the pressure from (1) and (2) yields

$$\beta^2\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = SM^2\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right),\ \beta^2 = 1 - M^2 \tag{7}$$

Taking account of (4) and (7) in the equation obtained by eliminating the pressure from (2) and (3) we find

$$Hv_y + S\Delta b_x = 0,\ H = \beta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{8}$$

From (5) and (4) we get

$$(\Delta - R\,\partial/\partial x)b_x + R\,\partial v_x/\partial y = 0 \tag{9}$$

Deriving (7) with respect to y and taking account of (4) and (9), we obtain

$$\left[\beta^2\frac{\partial}{\partial x}\left(\Delta - R\frac{\partial}{\partial x}\right) - RSM^2\Delta\right]b_x = R\frac{\partial^2 v_y}{\partial y^2} \tag{10}$$

Finally, from (8) and (10) we obtain

$$L\begin{pmatrix} v_y \\ b_x \end{pmatrix} = 0$$

$$L = H\frac{\partial}{\partial x}\left(\Delta - R\frac{\partial}{\partial x}\right) - RS\Delta\left(M^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \tag{11}$$

the operator L being the same as in previous papers.

3. Dispersion Equation

For plane waves of the form $\exp(-i\lambda x + sy)$, $s = -i\lambda r$, we get from (11) the following dispersion equation:

$$a(1 + r^2)^2 - b(1 + r^2) + c = 0$$

$$a = RS - i\lambda,\ b = R(1 + S + SM^2) - i\lambda M^2 \tag{12}$$

$$c = M^2R$$

The roots of Eq. (12) are distinct and have the imaginary part differing from zero. We denote by r_j ($j = 1,2$) those roots of Eq. (12) for which the real part of expressions $s_j = -i\lambda r_j$ are negative. This fact is possible since Eq. (12) is biquadratic such that two roots will surely enjoy this property indifferently if λ is positive or negative. The other two roots are of opposite sign. The roots of expression (12) are expressed by radicals.

Taking into account that if m and n are real numbers, we have

$$\left(m + \frac{n}{i\lambda}\right)^{1/2} = \begin{cases} (m)^{1/2} + \dfrac{n}{2(m)^{1/2}}\dfrac{1}{i\lambda} + 0(\lambda^{-2}),\ m > 0 \\ i(-m)^{1/2}\,\mathrm{sign}\lambda + \dfrac{n}{2(-m)^{1/2}}\dfrac{1}{|\lambda|} + 0(\lambda^{-2}),\ m < 0 \end{cases}$$

and we deduce that for large λ we have the following behavior:

$$r_1 = -\,i\,\mathrm{sign}\lambda + R0(|\lambda|^{-1})$$

$$r_2 = \begin{cases} -\,i\beta\,\mathrm{sign}\lambda + R0(|\lambda|^{-1}),\ \beta^2 > 0,\ \beta = (1 - M^2)^{1/2} \\ -(-\beta^2)^{1/2} + R0(\lambda^{-1}),\ \beta^2 < 0 \end{cases} \tag{13}$$

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